Indian Statistical Institute

Final Examination 2018-2019

Analysis II, B.Math First Year

Time : 3 Hours Date : 29.04.2019 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) X is a metric space. (iii) $B_r(a) = \{x \in X : d(x,a) < r\}$.

(1) (10 marks) Let $f : [0,1] \to \mathbb{R}$ be a bounded function, and suppose the lower integral of f on $[0,1]$ is positive. Prove that there exists an interval $[r_1, r_2] \subseteq [0,1]$, $r_1 \neq r_2$, such that $f(x) > 0$ for all x in $[r_1, r_2]$.

(2) (10 marks) Let f be a Riemann integrable function on $[a,b]$, $a < b$, and let $F(x) = \int_a^x f(t)dt$, $x \in [a,b]$. Prove that F is continuous on $[a,b]$.

(3) (10 marks) Find the limit (if exists): $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$.

(4) (10 marks) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^∞-function, and let

$$g(x_1, \ldots, x_n) = f(e^{x_1}, \ldots, e^{x_n}),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Suppose that

$$\sum_{i=1}^n \left(x_i^2 \frac{\partial^2 f}{\partial x_i^2} + x_i \frac{\partial f}{\partial x_i} \right) = 0.$$

Compute $\displaystyle\sum_{i=1}^n \frac{\partial^2 g}{\partial x_i^2}$.

(5) (15 marks) Which of the following statements are true, and which are false? Justify your answer.

(i) $\overline{B_r(a)} = \{x \in X : d(x,a) \leq r\}$.

(ii) If every subset of X is compact, then X is a finite set.

(iii) Interior of a connected subset of X is connected.

(6) (15 marks) Let X be a compact metric space, and let $f : X \to X$ be a function. Suppose that

$$d(f(x), f(y)) < d(x,y),$$

for all $x \neq y$. Prove that f has a unique fixed point.

(7) (15 marks) Let X be a complete countable metric space. Prove that there exists an element $x \in X$ such that $\{x\}$ is open.

(8) (15 marks) Determine the nature of the critical points of $f(x,y) = 2x^3 - 6xy + y^2 + 4y$, $(x,y) \in \mathbb{R}^2$.

(9) (15 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Suppose that f is differentiable at $(0,0)$ and

$$\lim_{x\to 0} \frac{f(x,x) - f(x,-x)}{x} = 1.$$

Compute $\frac{\partial f}{\partial y}(0,0)$.